Minimization Case

In certain cases it is difficult to obtain an initial basic feasible solution, such case arises;

When the constraints are of the ≤ type

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i,\ x_j \ge 0$$

but some right-hand side constants are negative ($b_i < 0$). In this case, after adding the non-negative slack variable S_i, the initial solution so obtained will be $S_i = -b_i$ for some i. It is not the feasible solution because it violates the non-negativity condition of slack variables.

When the constraints are of ≥ type

$$\sum_{j=1}^{n} a_{ij}x_j \ge b_i,\ x_j \ge 0$$

In this case to convert the inequalities into equation form, add surplus (neg- ative slack) variables

$$\sum_{j=1}^{n} a_{ij}x_j - S_i = b_i,\ x_j,\ S_i \ge 0$$

Letting $x_j = 0$, we get an initial solution $-S_i = b_i$ or $S_i = -b_i$. It is also not a feasible solution as it violates the non-negativity condition of surplus variables.

In this case we add artificial variables A_i to get an initial basic feasible solution. The resulting system of equations then becomes;

$$\sum_{j=1}^{n} a_{ij}x_j - S_i + A_i = b_i,\ x_j,\ S_i,\ A_i \ge 0, i = 1, 2, 3, \ldots, m$$

and has m equations and $(n+m+m)$ variables (i.e n-decision variables, m artificial variables and m surplus variables).

To get back to the original problem, artificial variables must be dropped out of the optimal solution. There are two methods for eliminating these variables from the solution

1. Two - Phase Method
2. Big-M Method or Method of Penalties.

The Two-Phase Method

In the first phase of this method the sum of all artificial variables is minimized subject to the given constraints to get a basic feasible solution of the LPP.

The second phase minimizes the original objective function starting with the basic feasible solution obtained at the end of the first phase. The steps of the algorithm are given bellow;

Phase I:

1. (a) If all the constraints in the given LPP are $\leq$ type then go to Phase
II. Otherwise, add some surplus and artificial variables to get equality constraints.
(b) If the given LPP is of minimization then convert to maximization.

2. Assign zero coefficients to each of the decision variables x_j and to the surplus variables and assign -1 coefficient to each of the artificial variables. This yields the following auxiliary LPP;

$$Max\ Z^* = \sum_{i=1}^{m} (-1)A_i$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j + A_i = b_i,\ x_j,\ A_i \geq 0, i = 1, 2, 3, \ldots, m$$

3. Apply the simplex algorithm to solve this auxiliary LPP. The following three cases may arise at optimality;

Max Z^* = 0 and atleast one artificial variable is present in the basis with positive value. Then no feasible solution exists for the original LPP.

Max Z^* = 0 and no artificial variable is present in the basis. Then the basis consists of only decision variables x_j's and hence we may move to Phase II to obtain an optimal basic feasible solution on the original LPP.

Max Z^* = 0 and atleast one artificial variable is present in the basis at zero value. Then a feasible solution to the above LPP is also a feasible solution to the original LPP. Now we may proceed direct to Phase II.

Phase II:

Assign actual coefficients to the variables in the objective function and zero to the artificial variables which appear at zero value in the basis at the end of Phase I. Then apply the usual simplex algorithm to the modified simplex table to get optimal solution to the original problem. Artificial variables which do not appear in the basis may be removed.

Example 1:Solve the following LP model using Two-Phase Method;

$$Max\ Z = 5x_1 - 4x_2 + 3x_3$$

Subject to

$$2x_1 + x_2 - 6x_3 = 20$$
$$6x_1 + 5x_2 + 10x_3 \leq 76$$
$$8x_1 - 3x_2 + 6x_3 \leq 50$$

and

$$x_1, x_2, x_3 \geq 0$$

Solution

After adding surplus variables S_1 and S_2 and artificial variable A_1 the problem becomes;

$$Max\ Z = 5x_1 - 4x_2 + 3x_3$$

subject to

$$2x_1 + x_2 - 6x_3 + A_1 = 20$$
$$6x_1 + 5x_2 + 10x_3 + S_1 = 76$$
$$8x_1 - 3x_2 + 6x_3 + S_2 = 50$$

and

$$x_1, x_2, x_3, S_1, S_2, A_1 \geq 0$$

Phase I:

Construction of Auxiliary LP model

$$Max\ Z^* = -A_1$$

subject to

$$2x_1 + x_2 - 6x_3 + A_1 = 20$$
$$6x_1 + 5x_2 + 10x_3 + S_1 = 76$$
$$8x_1 - 3x_2 + 6x_3 + S_2 = 50$$

and

$$x_1, x_2, x_3, S_1, S_2, A_1 \geq 0$$

Solution of an Auxiliary LP model

Table 1: Initial Solution

		$C_j \longrightarrow$	0	0	0	-1	0	0	
C_B	B	$b(=x_B)$	x_1	x_2	x_3	A_1	S_1	S_2	Min.Ratio
-1	A_1	20	2	1	-6	1	0	0	$\frac{20}{2} = 10$
0	S_1	76	6	5	10	0	1	0	$\frac{76}{6} = 12.66$
0	S_2	50	8	-3	6	0	0	1	$\frac{50}{8} = 6.25 \rightarrow$
$Z = -20$		$Z_j =$	-2	-1	6	-1	0	0	
		$C_j - Z_j$	2 $\uparrow$	1	-6	0	0	0	

Slack variable S_2 is removed from the basis since it has minimum ratio and variable x_1 is entering the basis since it has highest positive value into $C_j - Z_j$ row.

Iteration 1: The improved solution is obtained by performing the following elementary row operations.

$$R_3(new) \rightarrow \frac{R_3(old)}{8(key\,element)} = (25/4, 1, -3/8, 3/4, 0, 0, 1/8)$$

$$R_1(new) \rightarrow R_1(old) - (2)R_3(new) = (15/2, 0, 7/4, -15/2, 1, 0, -1/4)$$

$$R_2(new) \rightarrow R_2(old) - (6)R_3(new) = (77/2, 0, 29/4, 11/2, 0, 1, -3/4)$$

The improved solution is given in Table 2

Table 2: Improved Solution

		$C_j \longrightarrow$	0	0	0	-1	0	0	
C_B	B	$b(=x_B)$	x_1	x_2	x_3	A_1	S_1	S_2	Min.Ratio
-1	A_1	15/2	0	7/4	-15/2	1	0	-1/4	$\frac{15/2}{7/4} = 30/7 \rightarrow$
0	S_1	77/2	0	29/4	11/2	0	1	-3/4	$\frac{77/2}{29/4} = 154/29$
0	x_1	25/4	1	-3/8	3/4	0	0	1/8	-
$Z = -15/2$		$Z_j =$	0	-7/4	15/2	-1	0	1/4	
		$C_j - Z_j$	0	7/4 $\uparrow$	-15/2	0	0	-1/4	

Table 3: Improved Solution

		$C_j \longrightarrow$	0	0	0	-1	0	0
C_B	B	$b(=x_B)$	x_1	x_2	x_3	A_1	S_1	S_2
0	x_2	30/7	0	1	-30/7	4/7	0	-1/7
0	S_1	52/7	0	1	256/7	-29/7	1	2/7
0	x_1	55/7	1	0	-6/7	3/4	0	1/14
$Z = 0$		$Z_j =$	0	0	0	0	0	0
		$C_j - Z_j$	0	0	0	-1	0	0

Since all $C_j - Z_j \leq 0$ an optimal solution to the auxiliary LP model has been obtained and Max Z=0 with no artificial variable in the basis. item However, this solution may or may not be the basic feasible solution to the original LPP. Thus, go to Phase II to get an optimal solution to our original LPP.

Phase II

The modified simplex table from Table 3 is as follows;

Since all $C_j - Z_j \leq 0$ for all non-basic variables, the current basic feasible solution is also optimal. Hence, an optimum feasible solution to the given LPP is $x_1 = 55/7$, $x_2 = 30/7$, $x_3 = 0$, $S_1 = 52/7$, $S_2 = 0$, $S_3 = 0$ and Max. $Z = 155/7$.